REVIEW OF "THE DISTRIBUTION OF PRIME NUMBERS" BY DIMITRIS KOUKOULOPOULOS

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Why study prime numbers? Perhaps the immediate reasons are aesthetic. It is easy however to justify the topic after the fact. Prime numbers lead to some of the deepest structural open questions in mathematics, the Riemann Hypothesis and its generalizations. Their study also merges with our quest to understand the nature of randomness in mathematics.

The first result on primes is their infinitude, due to Euclid. More than a thousand years after Euclid, Dirichlet proved the infinitude of primes in arithmetic progressions $\{q\ell + a\}_{\ell > 0}$ for any fixed co-prime a and q. In Dirichlet's work we find the beginning of representation theory (Dirichlet characters), algebraic number theory (Dirichlet's class number formula) and analytic number theory (Dirichlet L-functions). Riemann took Dirichlet's work further and laid down a program to quantitatively understand the distribution of prime numbers both within the integers and in arithmetic progressions. Key to Riemann's program is the Riemann ζ-function defined as

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

for $\Re s > 1$. Riemann obtained the analytic continuation of $\zeta(s)$ to the entire complex plane and identified $\zeta(s)$ as a single function central to our understanding of the integers. On the one hand $\zeta(s)$ captures the fact that integers factor uniquely into prime numbers, this is reflected in the Euler product,

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}, \ \Re s > 1.$$

On the other hand $\zeta(s)$ captures the fact that the integers form a lattice, this is reflected (less obviously[1]) in the functional equation,

$$\xi(s) = \xi(1-s) \text{ where } \xi(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s).$$

Finally, Riemann showed that zeros of $\zeta(s)$ are in a sense dual objects to primes, and specifically if $\zeta(s)$ has no zeros in $\Re s > \frac{1}{2}$ then the primes are regularly distributed within the integers, or more precisely, as regularly as a generic random sequence can be. At the beginning of the 20th century this program is partially completed by de la Vallee Poussin and Hadamard with a proof of the prime number theorem,

$$\pi(x) := \{p \leq x : p \text{ prime}\} \sim \frac{x}{\log x}.$$

This classical material is discussed in various introductory books, e.g Ingham's "The distribution of prime number" [14] or Davenport's "Multiplicative number theory"

[1]The functional equation is equivalent to Poisson summation formula which uses fundamentally the fact that integers lie on a lattice

[1], or more recently in Montgomery and Vaughan's "Multiplicative number theory" [19] or Tenenbaum's "Analytic and Probabilistic number theory" [20]. This material is also presented in the first third (Part 1 & 2) of the book under review.

There is a heavy emphasis throughout standard textbooks on the connection between the Riemann zeta-function and the distribution of prime numbers. This creates a common impression that the Riemann ζ function and prime numbers are equivalent and interchangeable. Modern analytic number theory entirely dispells this idea; we currently have many results about primes that go far beyond anything that the Riemann zeta-function could ever deliver. One of the important aspect of the book under review is that it emphasizes the results on prime numbers and methods to study prime numbers that go beyond the reach of the Riemann zeta-function.

The rift away from the Riemann ζ-function starts in the third part of the book. Koukoulopoulos recognizes that in order to understand prime numbers one has to take a step back and first understand the integers, in particular the way in which integers factorize into prime numbers. The most appropriate concept for this task is that of a *multiplicative function*. Multiplicative functions are functions f that respect the multiplicative structure of the integers, that is $f(ab) = f(a)f(b)$ for integers a, b that are co-prime. As such they are particularly well suited for probing the ways in which integers factorize. The simplest question that one can ask about a multiplicative function f is the behavior of averages

$$\sum_{n \leq x} f(n).$$

as x goes to infinity. Such averages can then readily answer more direct questions about the multiplicative structure of the integers, for example questions about the number of prime factors of a "typical" integer.

Following Riemann, to understand such an average, it is natural to associate to a multiplicative function f a Dirichlet series $F(s) = \sum_{n \geq 1} f(n) n^{-s}$. For Riemann's method to succeed one needs to analytically continue F into the complex plane and then establish that F has desirable properties, such as a functional equation and an Euler product. Unfortunately this is possible only for a very restricted class of multiplicative functions (explicitely and still conjecturally described by the Langlands program). This approach misses many natural multiplicative functions, such as the Liouville function[2]. Moreover, it is often a tremendous achievement to obtain such an analytic continuation (e.g Wiles's proof of modularity of elliptic curves).

Therefore starting with the work of Wirsing [21], the field has seen the development of alternative methods for understanding averages of multiplicative functions. These methods work in great generality and do not demand nor yield excessive arithmetic information. Wirsing's method is based on setting up an integral delay equation connecting the average of a multiplicative function f on the primes with the average of f on the integers. Approximately solving the integral equation then yields the behavior of partial sums

$$\sum_{n \leq x} f(n).$$

[2]defined as 1 if n has an even number of prime factors and -1 otherwise

This is discussed in Chapter 14 of the book, and the earlier Chapter 13 showcases the more complex analytic “LSD method” which is more restrictive but yields stronger results.

The next two chapters 15 & 16 describe in a mere 15 pages many beautiful applications of mean-values of multiplicative functions. First the author proves the Erdös-Kac theorem according to which the number of prime factors of a typical integer n is normally distributed with mean and variance $\log \log n$. Then, he obtains information on the size of the jth smallest prime factor of a typical integer n: if $p_j(n)$ denotes the jth smallest prime factor of n then $\log \log p_j(n) \approx j$. Finally the author proves the Sathe-Selberg theorem which describes precisely the number of integers $n \leq x$ that have exactly k prime factors (with uniformity in k). These results are some of the highlights of probabilistic number theory, and a more in-depth treatment is contained in Elliott’s [2, 4] or Tenenbaum’s [20] books on the subject.

Equipped with the knowledge of the “anatomy of the integrs” the author introduces in Part 4 sieve methods. Sieves were invented by Viggo Brun in the 1910’s. In their most basic version sieves are designed to count the number of integers without small prime factors in various sequences. In the “combinatorial” incarnation a sieve is a variant of inclusion-exclusion inequalities that in addition incorporate information about how typical integers factor. Though elementary sieves are impressive; the most simple sieve immediately give results that are beyond the capability of the Riemann Hypothesis. For instance one of the earliest application of sieves was the proof that,

$$\sum_{p \text{ twin prime}} \frac{1}{p} < \infty.$$

The analogous series over all primes diverges. As an aside the comparison with the Riemann Hypothesis is not accidental. The Riemann Hypothesis (or the Generalized Riemann Hypothesis) is often used as a benchmark for the strength of a result in analytic number theory. To probe the strength of a result we often ask: does this result recover unconditionally a consequence of the Riemann Hypothesis? Does it go beyond what the Riemann Hypothesis can achieve?

Sieves also come in an “analytic” flavor first pioneered by Selberg. The principle is entirely different: we notice that if n has no small prime factors less than z then

$$1 \leq \Big(\sum_{\substack{d \mid n \\ d \leq z}} \lambda_d \Big)^2.$$

We then sum over all n of interest and optimize the choice of λ_d to minimize the resulting right-hand side (for example using Lagrange multipliers). While extremely elegant the Selberg sieve did not find until recently striking applications of its own (i.e that cannot be obtained with a combinatorial sieve). This however changed recently and dramatically with the work of Goldston-Pintz-Yildrim [8], Zhang [22] and Maynard(-Tao) [16] on the existence of small gaps between primes. This is discussed in the last part of the book.

An important result in sieve theory is the “fundamental lemma”. The fundamental lemma shows that sieve methods give us complete control over the small prime factors of integers, i.e prime factors of n that are less than $n^{\varepsilon(n)}$ for any $\varepsilon(n)$ that goes to 0 arbitrarily slowly. The author includes a very intuitive explanation of the proof of the fundamental lemma, explaining the meaning behind various technical parameter choices. He also concludes with a description of the more efficient β-sieve. A few standard applications are included (Brun-Titchmarsh, Titchmarsh divisor problem). Among the less standard ones is the very useful Shiu's bound for averages of multiplicative functions,

$$\sum_{x<n<x+x^{\delta}} f(n) \leq C_{\delta} x^{\delta} \exp\Big(\sum_{\substack{p \leq x \\ prime}} \frac{f(p)-1}{p} \Big).$$

with $C_{\delta} > 0$ a constant. Such bounds have in recent years found their way into far away areas; for example Shiu's bound is one of the crucial ingredients in the proof of the holomorphic quantum unique ergodicity conjecture by Holowinsky-Soundararajan [13]. The quantum unique ergodicity conjecture roughly concerns the behavior of eigenfunctions of the Laplacian in the high energy limits on arithmetic manifolds. Shiu's bound illustrates a point: knowledge of the anatomy of integers can be used to develop sieves, which are then fed back to yield even more precise results about the structure of the integers and percolate into far away applications. Such loops are common in the subject.

While this is not evident, the remaining two parts of the books are still concerned with sieve methods and their applications. The fifth part is called "Bilinear methods" but it could have aswell been called "analytic sieve methods". The author discusses here chiefly Vinogradov's method, one of the most powerful method for understanding primes using analytic techniques. Vinogradov's method can be described as a sieve method that expects highly non-trivial analytic information as an additional input. This method (in various more sophisticated variants and incarnations) forms the bedrock of most recent advances on prime numbers. For example it is crucial in the proof of the existence of primes of the form $x^4 + y^2$ by Friedlander-Iwaniec [6] (or of the form $x^3 + 2y^3$ by Heath-Brown [12]) or the proof by Maynard [18] of the existence of infinitely primes without the digit 7.

Let us briefly discuss the method. If we wish to understand the behavior of

$$\sum_{p \leq x} \kappa(p)$$

for some sequence κ that is far from being multiplicative then as a crucial ingredient Vinogradov's method expects bounds for so-called "bilinear sums",

$$\sum \alpha_n \beta_m \kappa(mn)$$

where the sequences α_n and β_m are arbitrary. It is bewildering that a sum of such generality can be bounded non-trivially; this is possible, at least in principle, because the sequence κ does not have multiplicative structure. In practice non-trivial bounds can be obtained rigorously by bounding the operator norm of a matrix with entries $\kappa(mn)$. Interestingly a similar method in harmonic analysis, discovered significantly later, goes by the name of the $TT^{\star}$ method (having to do with the fact that the operator norm of $TT^{\star}$ and $T^{\star}T$ matches).

The author includes as an application of Vinogradov's method a result on counting arithmetic progressions of length three in the primes. This is the first non-trivial case of the Green-Tao theorem [10], even though the methods used by Green-Tao for longer progressions are entirely different.

The last chapter of this part of the book discusses Linnik's theorem. Linnik's theorem asserts that we don't have to "wait too long" before seeing a prime in an arithmetic progression. Specifically there exists an $A > 1$ and $C > 1$ such that for any $(a, q) = 1$ there exists a prime with $p \equiv a \pmod q$ *and* $p \leq Cq^A$ (see [11] for a result with $A = 5.5$). Linnik's theorem recovers a consequence of the Generalized Riemann Hypothesis. Until recently it was considered an extraordinality deep result, using the full scope of the machinery of sieves and L-functions. In recent times the proof has been very significantly streamlined by Elliott [3], Friedlander-Iwaniec [7] and subsequently Granville-Soundararajan-Harper [9]. The author includes here a proof along these lines. The proof that is presented has not appeared before in book form and forms one of the highlights of the book.

The last and most delightful part of the book illustrates two recent major breakthrough about primes; specifically the author discusses Maynard's [16] proof of Zhang's [22] theorem on the existence of bounded gaps between primes and the recent work of Maynard [17] and Ford-Green-Konyagin-Tao [5] on the existence of large gaps. The last part serves as an illustration of how much can be achieved just by using a subset of the methods discussed in the book. Hopefully this will motivate the reader to plow further into the subject!

In summary this is an excellent book introducing the reader to a wealth of modern techniques for studying prime numbers. There is a lot of new material here that has never appeared before in book form. The author took great care in explaining both the intuition behind this very technical subject and in providing the "best" proofs, especially proofs that are short and understandable. The book will be an excellent introduction to anybody interested in primes at a research level (or rather interested in reaching quickly this level). It is however not a terminus, the serious reader will then want to look at follow-up books such as for example "Analytic number theory" by Iwaniec-Kowalski [15] or "Opera de Cribro" by Friedlander-Iwaniec [7] which introduces additional techniques, for example automorphic forms, the circle method or more complex sieve or harmonic analytic methods.

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